Teoria Espectral

1. Adjoint. Symmetric and Self-Adjoint Operators

We assume that $\mathcal{H}$ is a Hilbert space. In this section we will define self-adjoint unbounded operators and study its properties.

Definition 1.1. *Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a linear operator densely defined (d.d.) i.e. $\overline{D(A)} = \mathcal{H}$.*

We define the **adjoint** *A^* of the operator A by*

$$\begin{cases} D(A^*) = \{\eta \in \mathcal{H} : \exists \psi \in \mathcal{H} \text{ such that } (A\phi, \eta) = (\phi, \psi)\ \forall \phi \in D(A)\}, \\ A^*\eta = \psi. \end{cases}$$

Remarks 1.2.

(1) *A^* is well defined.*

If there exists $\tilde{\psi} \in \mathcal{H}$ such that

$$(A\phi, \eta) = (\phi, \psi) = (\phi, \tilde{\psi})$$

Then

$$(\phi, \psi - \tilde{\psi}) = 0 \text{ for all } \phi \in D(A)$$

which implies that $\psi \equiv \tilde{\psi}$ since $D(A)$ is dense in $\mathcal{H}$.

(2) *We have that*

$$D(A^*) = \{\eta \in \mathcal{H} :\ L\eta : \phi \in D(A) \mapsto (A\phi, \eta) \text{ is continuous}\}. \tag{1.1}$$

Indeed, we can extend $\overline{L\eta} : \overline{D(A)} = \mathcal{H} \to \mathbb{C}$ continuous. i.e. $L\eta \in \mathcal{H}^*$ implies there exists $\psi \in \mathcal{H}$ such that

$$\overline{L\eta}(\phi) = (\phi, \psi)\ \forall \phi \in \mathcal{H}$$

by the Riesz Theorem.

In particular, it holds that

$$(A\phi, \eta) = (\phi, \psi) \quad \forall \phi \in D(A).$$

(3) *It holds that*

$$(A\phi, \eta) = (\phi, A^*\eta), \quad \forall \phi \in D(A),\ \forall \eta \in D(A^*). \tag{1.2}$$

Verify that $A^* : D(A^*) \subset \mathcal{H} \to \mathcal{H}$ defines a linear operator.

Exercise 1.3. *If $A \in \mathcal{B}(\mathcal{H})$, show that $A^* \in \mathcal{B}(\mathcal{H})$ and it holds that*

$$(Af, g) = (f, A^*g) \quad \forall f, g \in \mathcal{H}$$

and

$$\|A\| = \|A^*\|.$$

Properties. Let $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ and $B : D(B) \subseteq \mathcal{H} \to \mathcal{H}$ be two linear operators d.d. Then it holds that

(i) A^* is closed.
(ii) $(\lambda A)^* = \bar{\lambda} A^*$ for all $\lambda \in \mathbb{C}$.
(iii) $A \subseteq B$ implies that $B^* \subseteq A^*$.
(iv) $A^* + B^* \subseteq (A + B)^*$.
(v) $B^* A^* \subseteq (AB)^*$.
(vi) $A \subseteq A^{**}$ where $A^{**} = (A^*)^*$.
(vii) $(A + \lambda)^* = A^* + \bar{\lambda}$.

Theorem 1.4. *Let $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ a linear operator d.d., then it holds that*

(i) *A^* is closed;*
(ii) *A is closable if and only if A^* is d.d. in this case $\bar{A} = A^{**}$;*
(iii) *If A is closable, then $(\bar{A})^* = A^*$.*

Proof. To prove this theorem we will need some definitions and lemmas.

We start by considering the Hilbert space $\mathcal{H} \times \mathcal{H}$ equipped with the inner product

$$\langle (\phi_1, \psi_1), (\phi_2, \psi_2) \rangle = (\phi_1, \phi_2)_{\mathcal{H}} + (\psi_1, \psi_2)_{\mathcal{H}}.$$

We define the operator

$$V : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$$
$$(\phi, \psi) \mapsto V(\phi, \psi) = (-\psi, \phi).$$

Notice that V is a unitary operator. In fact,

$$\begin{aligned} \langle V(\phi_1, \psi_1), V(\phi_2, \psi_2) \rangle &= \langle (-\psi_1, \phi_1), (-\psi_2, \phi_2) \rangle \\ &= (\psi_1, \psi_2) + (\phi_1, \phi_2) \\ &= \langle (\phi_1, \psi_1), (\phi_2, \psi_2) \rangle. \end{aligned}$$

Lemma 1.5. *If $V : \mathcal{H} \to \mathcal{H}$ is a unitary operator, then*

$$V(E^\perp) = V(E)^\perp, \quad \forall E \subseteq \mathcal{H},$$

where $E^\perp$ denotes the orthogonal set to E which is defined as

$$E^\perp = \{\phi \in \mathcal{H} : (\phi, \eta) = 0, \ \forall \eta \in E\}.$$

Proof. Let $x \in E^{\perp}$ and $y = V(e) \in V(E)$, then

$$\langle V(x), y\rangle = \langle V(x), V(e)\rangle = \langle x, e\rangle = 0.$$

Thus $V(x) \in V(E)^{\perp}$ and so $V(E^{\perp}) \subset V(E)^{\perp}$.

Reciprocally, let $y \in V(E)^{\perp}$, since V is a bijection it implies there exists a unique $x \in \mathcal{H}$ such that $y = V(x)$.

If for all $e \in E$,

$$\langle x, e\rangle = \langle V(x), V(e)\rangle = 0$$

this implies that $x \in E^{\perp}$ and so $y \in V(E^{\perp})$. Thus $V(E)^{\perp} \subset V(E^{\perp})$. This concludes the proof of the lemma. □

Proof of (i) We denote by

$$G(A) = \{(x, Ax) : x \in D(A)\} \subseteq \mathcal{H} \times \mathcal{H}.$$

the graph of the operator A.

We can see that

$$\begin{aligned}
(\phi,\eta) \in V(G(A))^{\perp} &\iff \langle(\phi,\eta),(-A\psi,\psi)\rangle = 0 \quad \forall\psi \in D(A)\\
&\iff -(\phi, A\psi) + (\eta,\psi) = 0 \quad \forall\psi \in D(A)\\
&\iff (\phi, A\psi) = (\eta,\psi) \quad \forall\psi \in D(A)\\
&\iff (A\psi, \phi) = (\psi,\eta) \quad \forall\psi \in D(A)\\
&\iff (\phi,\eta) \in G(A^*).
\end{aligned}$$

This shows that $G(A^*) = \{V(G(A))\}^{\perp}$ is closed. (since the orthogonal of a set is always a closed subspace). Thus A^* is a closed operator.

Lemma 1.6. *If $E \subseteq \mathcal{H}$, then $\overline{E} = (E^{\perp})^{\perp}$.*

Proof. If $e \in E$ and $x \in E^{\perp}$, then $(e, x) = 0$ and so $e \in (E^{\perp})^{\perp}$. Hence $E \subset (E^{\perp})^{\perp}$. Therefore $\overline{E} \subset (E^{\perp})^{\perp}$.

Now we know that $\mathcal{H} = \overline{E} \oplus (\overline{E})^{\perp}$ because of the orthogonal projection Theorem.

On the other hand, $y \in (\overline{E})^{\perp}$ implies that $(y, x) = 0$ for all $x \in \overline{E}$ and this means that $y \in E^{\perp}$. Similarly, $y \in E^{\perp}$ implies that $(y, x) = 0$ for all $x \in E$ and so we have (y, x)=0 for all $x \in (\overline{E})^{\perp}$. We deduce from this that $(\overline{E})^{\perp} = E^{\perp}$.

Thus

$$\mathcal{H} = \overline{E} \oplus E^{\perp} = E^{\perp} \oplus (E^{\perp})^{\perp}.$$

We conclude that $\overline{E} = (E^{\perp})^{\perp}$. □

Remark 1.7. *From the orthogonal projection Theorem we deduce that for any* $F \subset \mathcal{H}$ *closed and for all* $x \in \mathcal{H}$*, there exists a unique* $y = \mathcal{P}_F(x) \in F$ *such that* $(x - \mathcal{P}_F(x), z) = 0$ *for all* $z \in F$*, then* $x - \mathcal{P}_F(x) \in F^{\perp}$*, where* $\mathcal{P}_F(x)$ *denotes the projection on* F *at* x*.*

Since $G(A) \subset \mathcal{H} \times \mathcal{H}$, we have from Lemma 1.6, Lemma 1.5 and the fact that V is unitary that

$$\begin{aligned}\overline{G(A)} = \{G(A)^{\perp}\}^{\perp} &= \{V^2(G(A)^{\perp})\}^{\perp} \\ &= \{V[V(G(A)^{\perp})]\}^{\perp} = \{V[G(A^*)]\}^{\perp}.\end{aligned}$$

Thus,

$$\overline{G(A)} = \{V[G(A^*)]\}^{\perp} \tag{1.3}$$

and

$$G(A^*) = \{V[G(A)]\}^{\perp} \tag{1.4}$$

This means that A^* d.d. implies that A is closable.

Proof of (ii). If A^* is d.d then we deduce from (1.4) and (1.3) that

$$\{V[G(A^*)]\}^{\perp} = G(A^{**}) = \overline{G(A)}.$$

Then A is closable and $\overline{A} = A^{**}$.

Reciprocally, if $D(A^*)$ were not dense in $\mathcal{H}$, let $\psi \in D(A^*)^{\perp}$, $\psi \neq 0$, then $(\psi, 0) \in G(A^*)^{\perp}$ which implies that $(0, \psi) \in \{VG(A^*)\}^{\perp} = \overline{G(A)}$ but by Lemma 1.17 in [4] we have that $\overline{G(A)}$ is not the graph of any linear operator, that is, A is not a closable operator. This implies (ii).

Proof of (iii). Let A be a closable operator, since A^* is closed and (ii) holds we deduce that

$$A^* = \overline{A^*} = A^{***} = (A^{**})^* = (\overline{A})^*.$$

This completes the proof of Theorem 1.4. □

Example 1.8. *We consider once again the operator* A *defined by*

$$\begin{cases} D(A) = C^0([0,1]) \subset L^2([0,1]) \to L^2([0,1]) \\ Af = \phi\, f(1) \quad \text{where } \phi \in L^2([0,1]),\ \phi \neq 0. \end{cases}$$

In Example 1.21 in [4] *we saw that the operator* A *is not closable. We shall show now that* A^* *is not densely defined.*

Let $\eta \in \mathcal{H} = L^2([0,1])$*, then*

$$(Af, \eta) = \int_0^1 f(1)\phi(x)\overline{\eta}(x)\, dx$$

Observe that the map

$$f \in C([0,1]) \subset L^2([0,1]) \mapsto f(1) \in \mathbb{C}$$

is not continuous in the L^2 topology. (To see this, take for instance $f_n(x) = x^n$. It is easy to check that $f_n \overset{L^2}{\to} 0$ and $f_n(1) = 1$.) As a consequence the only choice in order to have the map $f \mapsto (Af, \eta)$ continuous is taking $\eta \in L^2([0,1])$ such that

$$\int_0^1 \phi(x)\overline{\eta}(x)\, dx = 0.$$

But this implies that

$$\begin{cases} D(A^*) = \{\phi\}^\perp \\ A^*\eta = 0. \end{cases}$$

In particular $\{\phi\}^\perp$ is not dense in $L^2([0,1])$.

1.1. **Application. Differentiable operators are closable.** Consider a differentiable operator of order m,

$$P(x,D) = \sum_{\substack{|\alpha|\le m \\ \alpha\in\mathbb{N}^n}} a_\alpha(x) D_x^\alpha$$

where $a_\alpha \in C^\infty(\Omega)$ and $\Omega \subseteq \mathbb{R}^n$ is an open set.

We can define $P_{\min}$ by

$$\begin{cases} D(P_{\min}) = C_0^\infty(\Omega) \subseteq L^2(\Omega) \\ P_{\min}\phi = P(x,D)\phi. \end{cases}$$

Taking $\phi, \psi \in C_0^\infty(\Omega)$, then

$$\begin{aligned} (P_{\min}\phi, \psi)_0 &= (\sum_{|\alpha|\le m} a_\alpha(x)\partial_x^\alpha \phi, \psi)_0 \\ &= \int_\Omega \sum_{|\alpha|\le m} a_\alpha(x)\partial_x^\alpha\phi(x)\overline{\psi(x)}\, dx \\ &= \sum_{|\alpha|\le m} \int_\Omega a_\alpha(x)\partial_x^\alpha\phi(x)\overline{\psi(x)}\, dx \\ &= \sum_{|\alpha|\le m} (-1)^{|\alpha|} \int_\Omega \phi(x)\overline{\partial_x^\alpha(a_\alpha(x)\psi(x))}\, dx. \end{aligned}$$

This gives us

$$(P_{\min}\phi, \psi)_0 = (\phi, \zeta)_0$$

where $\zeta = \sum_{|\alpha|\leq m} (-1)^{|\alpha|}\partial_x^\alpha(\overline{a_\alpha}\,\psi)$. Hence defining $P^*_{\min}$ as

$$\begin{cases} D(P^*_{\min}) = C_0^\infty(\Omega), \\ P^*_{\min}\psi = \sum_{|\alpha|\leq m} (-1)^{|\alpha|}\partial_x^\alpha(\overline{a_\alpha}\,\psi). \end{cases}$$

This implies that $P^*_{\min}$ is densely defined and from Theorem 1.4 it follows that $P_{\min}$ is a closable operator.

1.2. **Symmetric and Self-Adjoint Operators.**

Definition 1.9. *Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a linear operator.*

(i) *We say that A is a* **symmetric operator** *if*

$$(A\phi, \psi) = (\phi, A\psi), \quad \text{for all } \phi, \psi \in D(A).$$

This is equivalent to say that $A \subseteq A^*$.

(ii) *We say that A is a* **self-adoint operator** *if $A = A^*$.*

Exercise 1.10. *We say that a linear operator A is a* **maximal symmetric** *operator if*

$$A \subseteq A^*$$

and

$$A \subseteq B,\ B \subseteq A^* \quad \text{then } A = B.$$

Prove that if $A = A^*$ then A is maximal symmetric.

Example 1.11. *The operator H_0 is symmetric, i.e. $H_0 \subseteq H_0^*$. We recall that*

$$\begin{cases} D(H_0) = H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \\ H_0 f = -\Delta f \end{cases}$$

For all $f, g \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned} (H_0 f, g)_0 &= -\int_{\mathbb{R}^n} \Delta f(x)\,\overline{g(x)}\,dx \\ &= -\sum_{j=1}^n \int_{\mathbb{R}^n} \partial_{x_j}^2 f(x)\,\overline{g(x)}\,dx \\ &= -\sum_{j=1}^n \int_{\mathbb{R}^n} f(x)\,\partial_{x_j}^2\overline{g(x)}\,dx. \end{aligned}$$

Then

$$(H_0 f, g)_0 = (f, H_0 g)_0 \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n).$$

Now let $f, g \in H^2(\mathbb{R}^n)$, *there exist* $\{f_j\}$, $\{g_j\} \subseteq \mathcal{S}(\mathbb{R}^n)$ *such that* $f_j \overset{H^2}{\to} f$ *and* $g_j \overset{H^2}{\to} g$.

Notice that $\overline{C_0^\infty(\mathbb{R}^n)}^{H^2} = L^2(\mathbb{R}^n)$ *(prove it!). Thus for any* $j \in \mathbb{N}$ *it holds that*

$$(H_0 f_j, g_j)_0 = (f_j, H_0 g_j)_0$$

and making $j \to \infty$ *it follows that*

$$(H_0 f, g)_0 = (f, H_0 g)_0.$$

From the last identity we deduce that $H^2(\mathbb{R}^2) = D(H_0) \subseteq D(H_0^*)$ *and* $H_0 \subseteq H_0^*$.

Is H_0 a self-adjoint operator?

We need the following definition

Definition 1.12. *Let* $k \in \mathbb{N}$, $1 \leq p \leq \infty$. *Given a domain* $\Omega \subset \mathbb{R}^n$, *the* **Sobolev space** $W^{k,p}(\Omega)$ *is defined as,*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega),\ \forall |\alpha| \leq k\}.$$

We equipped the Sobolev space with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \Big(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \Big)^{\frac{1}{p}}, & 1 \leq p < \infty. \\ \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}, & p = \infty. \end{cases}$$

It is usual to denote $W^{k,2}(\Omega)$ *by* $H^k(\Omega)$ *for it is a Hilbert space with the norm* $W^{k,2}(\Omega)$.

Example 1.13. *Consider the operators* $A_j = \dfrac{1}{i}\dfrac{d}{dx}$, $j = 0, 1, 2$ *with*

$$\begin{aligned} D(A_0) &= H^1([-\pi, \pi]) \subseteq L^2([-\pi, \pi]); \\ D(A_1) &= \{\phi \in H^1([-\pi, \pi]) : \phi(-\pi) = \phi(\pi)\}; \\ D(A_2) &= \{\phi \in H^1([-\pi, \pi]) : \phi(-\pi) = \phi(\pi) = 0\}. \end{aligned}$$

We will see that

(i) A_1 *and* A_2 *are symmetric operators.*
(ii) A_0 *is not a symmetric operator.*
(iii) $A_2 = A_0^*$ $(\subseteq A_0)$ *this implies that* $A_2^* = A_0^{**} = \overline{A_0} = A_0$ *but* $A_0 \supsetneq A_2$ *and so these operators are not self-adjoints.*

(iv) $A_1^* = A_1$.

Proof of (i). Let $\phi, \psi \in D(A_0)$ then

$$\begin{aligned}(A_0\,\phi, \psi) &= (\frac{1}{i}\phi', \psi)_{L^2} = \frac{1}{i}\int_{-\pi}^{\pi} \phi'(x)\,\overline{\psi(x)}\,dx \\ &= \frac{1}{i}\phi(x)\,\overline{\psi(x)}\,\Big|_{-\pi}^{\pi} - \frac{1}{i}\int_{-\pi}^{\pi} \phi(x)\,\overline{\psi'(x)}\,dx \\ &= \frac{1}{i}\big[\phi(\pi)\,\overline{\psi}(\pi) - \phi(-\pi)\,\overline{\psi}(-\pi)\big] - \frac{1}{i}\int_{-\pi}^{\pi} \phi(x)\,\overline{\psi'(x)}\,dx\end{aligned}$$

This holds for absolutely continuous functions. Then

$$(A_0\,\phi, \psi) = \frac{1}{i}\big[\phi(\pi)\,\overline{\psi}(\pi) - \phi(-\pi)\,\overline{\psi}(-\pi)\big] + (\phi, A_0\,\psi)_{L^2} \tag{1.5}$$

for all $\phi, \psi \in H^1([-\pi, \pi])$.

Therefore if $\phi, \psi \in D(A_j), j = 1, 2$, we deduce that

$$(A_j\,\phi, \psi)_{L^2} = (\phi, A_j\psi)_{L^2} \quad j = 1, 2,$$

which implies that $A_1 \subseteq A_1^*$ and $A_2 \subseteq A_2^*$.

Proof of (ii). In addition, there exist $\phi, \psi \in H^1([-\pi, \pi])$ such that

$$\phi(\pi)\,\overline{\psi}(\pi) \neq \phi(-\pi)\,\overline{\psi}(-\pi)$$

it yields

$$(A_0\,\phi, \psi)_{L^2} \neq (\phi, A_0\,\psi)$$

which implies that $A_0 \subsetneq A_0^*$.

Proof of (iii). Now let $\phi \in D(A_0)$ and $\psi \in D(A_2)$, then from (1.5) it follows that

$$(A_0\phi, \psi) = (\phi, A_2\psi)$$

which implies that $A_2 \subseteq A_0^*$.

To prove that $A_0^* \subseteq A_2$ it is enough to verify $D(A_0^*) \subseteq D(A_2)$.

Claim. If $\eta \in D(A_0^*)$, then

$$\int_{-\pi}^{\pi} A_0^*\eta(y)\,dy = 0. \tag{1.6}$$

Indeed, since $1 \in H^1([-\pi, \pi])$, it follows that

$$\int_{-\pi}^{\pi} \overline{A_0^*\eta(y)}\,dy = (1, A_0^*\eta) = (A_0(1), \eta) = 0.$$

Let $\eta \in D(A_0^*)$, we define $w(x) = i\int_{-\pi}^{x} A_0^*\eta(y)\,dy$. Then $w(-\pi) = w(\pi) = 0$ and so $w \in D(A_2)$ (prove it!).

Moreover,

$$A_2 w = \frac{1}{i} w'(x) = A_0^* \eta(x)$$

But since $A_2 \subseteq A_0^*$ *we have* $A_0^* w = A_2 w$*. Thus*

$$\begin{aligned} 0 &= (\phi, A_0^*(w - \eta)) \\ &= (A_0 \phi, w - \eta) \quad \forall \phi \in D(A_0). \end{aligned} \tag{1.7}$$

This implies that $w = \eta \in D(A_2)$ *and so* $D(A_0^*) \subseteq D(A_2)$*.*

To deduce the last affimation we have used that

$$C_0^\infty([-\pi, \pi]) \subseteq \overline{(R(A_0))} = L^2([-\pi, \pi])$$

then taking $u \in C_0^\infty([-\pi, \pi])$ *we have that* $v(x) = \int_{-\pi}^{x} u(y)\, dy \in D(A_0)$ *and* $A_0 v = u$*.*

Proof of (iv). *Notice that* $A_2 = A_0^* \subseteq A_1 \subseteq A_0$*. Then* $A_0^* \subseteq A_1^* \subseteq A_0^{**} = \overline{A_0} = A_0$*. Thus for all* $\psi \in D(A_1^*)$*,*

$$A_1^* \psi = \frac{1}{i} \psi'.$$

From (ii) we already know that $A_1 \subseteq A_1^*$*, we need to show now that* $D(A_1^*) \subseteq D(A_1)$*.*

By the identity (1.5) *it follows that for all* $\phi \in D(A_1)$*, and for all* $\psi \in D(A_1^*)$*,*

$$(A_1 \phi, \psi)_{L^2} = (\phi, A_1^* \psi)$$

since $A_1^* \subseteq A_0$ *and*

$$\phi(\pi)\overline{\psi}(\pi) - \phi(-\pi)\overline{\psi}(-\pi) = \phi(\pi)(\overline{\psi}(\pi) - \overline{\psi}(-\pi)).$$

This implies that

$$\phi(\pi)(\overline{\psi}(\pi) - \overline{\psi}(-\pi)) = 0.$$

Choosing $\phi \equiv 1 \in D(A_1)$ *it follows that* $\psi(\pi) = \psi(-\pi)$*, that is,* $\psi \in D(A_1)$ *that concludes the proof.*

Remark 1.14. *From this example we can see that it is not easy at all to establish when a linear symmetric operator is self-adjoint just by using the definition. In what follows we will establish a criteria to determine when a symmetric operator is self-adjoint.*

1.3. **Basic Criteria.** The next result is an effective tool to determine when a symmetric operator is sefl-adjoint.

Theorem 1.15. *Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ a linear operator d.d. such that $A \subseteq A^*$, then the following assertions are equivalent:*

(i) $A = A^*$;
(ii) A *is closed and* $Ker(A^* \pm i) = \{0\}$;
(iii) $R(A \pm i) = \mathcal{H}$.

Remarks 1.16.

(1) *The linear operator A does have any special feature, i.e. the criteria holds for $\pm\lambda i$, $\lambda > 0$.*
(2) *To prove* (iii) $\implies$ (ii) *it is necessary to have $R(A + i) = \mathcal{H}$ and $R(A - i) = \mathcal{H}$.*

Proof.
(i) $\implies$ (ii). If $A = A^*$, then A is closed.
Let now $\phi \in D(A^*) = D(A)$ such that $A^*\phi = i\,\phi$. Then

$$i\|\phi\|^2 = (i\phi, \phi) = (A^*\phi, \phi) = (A\phi, \phi) = (\phi, A^*\phi) = (\phi, i\phi) = -i\|\phi\|^2.$$

This implies that $\phi = 0$, that is, $Ker(A^* - i) = \{0\}$.

Similarly, we will have $Ker(A^* + i) = \{0\}$.

(ii) $\implies$ (iii). We will follow the following strategy:

(1) We first prove that $R(A \pm i)^\perp = \{0\}$.
(2) Then we show that $R(A \pm i)$ is closed.

Thus we can conclude that $R(A \pm i) = \mathcal{H}$. The latter follows from the orthogonal projection Theorem.

Affirmation. Let $B : D(B) \subset \mathcal{H} \to \mathcal{H}$ be a linear operator d.d., then

$$R(B)^\perp = KerB^*.$$

Proof of the Affirmation. ($\subseteq$) Let $\phi \in R(B)^\perp$, then for all $\psi \in D(B)$ we have that $(B\psi, \phi) = 0$. It follows that $\phi \in D(B^*)$ and $B^*\phi = 0$ which implies that $\phi \in KerB^*$.

($\supseteq$) Let $\phi \in KerB^*$ then for all $\psi \in D(B)$

$$(B\psi, \phi) = (\psi, B^*\phi) = 0$$

we conclude that $\phi \in R(B)^\perp$. This completes the proof of the affirmation. □

Now we suppose (ii), we deduce from the affirmation that

$$R(A \pm i)^{\perp} = Ker(A \pm i)^{*} = Ker(A^{*} \mp i) = \{0\}$$

where we use property (vii) in page 2. The identity above gives us (1).

Next we shall show (2), i.e. $R(A \pm i)$ is closed.

Let $\{f_j\} \subset R(A \pm i)$ such that $f_j \overset{\mathcal{H}}{\to} f$. For all j there exist ϕ_j such that $f_j = (A \pm i)\phi_j$.

On the other hand, for all $\phi \in D(A)$ we have that

$$\begin{aligned}\|(A \pm i)\phi\|^2 &= \big((A \pm i)\phi, (A \pm i)\phi\big) \\ &= (A\phi, A\phi) \pm i(\phi, A\phi) \mp i(\phi, A\phi) - i^2(\phi, \phi).\end{aligned}$$

Using that $A \subseteq A^*$ we conclude that

$$\|(A \pm i)\phi\|^2 = \|A\phi\|^2 + \|\phi\|^2. \tag{1.8}$$

From (1.8) we have that

$$\|f_j - f_l\|^2 = \|A(\phi_j - \phi_l)\|^2 + \|\phi_j - \phi_l\|^2 \quad \forall j, l. \tag{1.9}$$

Then we deduce from (1.9) that

$$\begin{cases} \|\phi_j - \phi_l\| \leq \|f_j - f_l\| \underset{j,l\to\infty}{\to} 0, \\ \|A(\phi_j - \phi_l)\| \leq \|f_j - f_l\| \underset{j,l\to\infty}{\to} 0. \end{cases}$$

Thus $\{\phi_j\}$ and $\{A\phi_j\}$ are Cauchy sequences in $\mathcal{H}$. Since $\mathcal{H}$ is complete it follows that there exist $\phi, \psi \in \mathcal{H}$ such that

$$\begin{cases} \phi_j \overset{\mathcal{H}}{\to} \phi \\ A\phi_j \overset{\mathcal{H}}{\to} \psi \end{cases}$$

as $j \to \infty$.

Since A is closed it follows that $\phi \in D(A)$ and $\psi = A\phi$.

Thus

$$(A \pm i)\phi_j \underset{j\to\infty}{\to} (A \pm i)\phi \in R(A \pm i).$$

Therefore $R(A \pm i)$ is closed.

<u>(iii) $\Longrightarrow$ (i).</u> We already know that $A \subseteq A^*$. Left to show that $D(A^*) \subseteq D(A)$.

Let $f \in D(A^*)$, by hypothesis $R(A - i) = \mathcal{H}$. Hence there exists $\phi \in D(A)$ such that

$$(A^* - i)f = (A - i)\phi \underset{A\subseteq A^*}{=} (A^* - i)\phi.$$

From this we conclude that

$$f - \phi \in Ker(A^* - i) = Ker(A + i)^* = R(A + i)^{\perp} = \{0\}$$

Thus $f = \phi \in D(A)$. Above we used the affirmation and the fact that $R(A + i) = \mathcal{H}$. $\square$

Corollary 1.17 (Spectrum of a self-adjoint operator)**.** *If $A = A^*$, then*

$$\sigma(A) \subseteq \mathbb{R}.$$

Proof. The basic criteria implies that for all $\lambda > 0$,

$$\begin{cases} Ker(A \pm i\lambda) = \{0\} \\ R(A \pm i\lambda) = \mathcal{H}. \end{cases}$$

From this we conclude that $\pm i\lambda \in \rho(A)$ for all $\lambda > 0$. If we denote $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$, then $i\mathbb{R}^* \subseteq \rho(A)$.

Moreover, for all $\eta > 0$ we obtain

$$(A + \eta)^* = A^* + \overline{\eta} = A^* + \eta.$$

From this it follows then that

$$\begin{cases} Ker(A + \eta \pm i\lambda) = \{0\}, \\ R(A + \eta \pm i\lambda) = \mathcal{H}. \end{cases}$$

Hence $\eta \pm i\lambda \in \rho(A)$, for all $\eta \in \mathbb{R}$ and $\lambda > 0$. Therefore $\mathbb{C}\backslash\mathbb{R} \subset \rho(A)$. In other words, $\sigma(A) \subset \mathbb{R}$. $\square$

Definition 1.18. *A linear symmetric operator ($A \subseteq A^*$) is called an* **essentially self-adjoint** *operator, if and only if, $\overline{A}$ is self-adjoint. That is, $\overline{A}^* = A^* = \overline{A}$.*

We can state another version of the Basic Criteria with the same proof as follows.

Theorem 1.19. *Let A be a symmetric operator ($A \subseteq A^*$), then the following statements are equivalent.*

(i) $\overline{A} = A^*$ *(A is essentially self-adjoint);*
(ii) $Ker(A^* \pm i) = \{0\}$*;*
(iii) $(A^* \pm i)$ *are dense in $\mathcal{H}$.*

Example 1.20. *The operator $H_{0_{\min}}$ is essentially self-adjoint (prove it!).*

References

[1] M. Reed and B. Simon, Methods of Modern Mathematical Physics, Volumes 1, 2
[2] E. Hille, Methods in Classical and Functional Analysis
[3] T. Kato, Perturbation Theory
[4] Notes on Unbounded Operators.